A BRIEF INTRODUCTION TO THE SCHRAMM-LOEWNER EVOLUTION

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Abstract. In this article, we expose the reader to the theory of the Schramm Loewner evolution and provide intuition. We will motivate it as the scaling limit of percolation on a hexagonal grid and show that the Schramm Loewner evolution is the curve with the properties we would expect from such a limit. We also mention basic properties and related open problems.

1. Introduction

The Schramm Loewner evolution is (roughly) the family of random non-self-crossing curves which satisfy conformal invariance and a "domain Markov property." These turn out to be the scaling limits of several statistical physics models and have also found applications in studying the geometry of Brownian motion, see for example [MP01] for an introduction to this topic. In what follows, we take critical percolation on a grid as our motivating example and discuss how the Schramm Loewner evolution arises naturally.

1.1. **Critical Percolation.** Consider some simply connected domain (open and connected) D of the complex plane $\mathbb{C}$ bounded by a Jordan curve. Fix two points $a, b \in \partial D$ and let A denote an arc from a to b. Fix some sufficiently small $\delta > 0$ and let $\mathcal{H}_\delta$ denote a planar hexagonal grid with edge length δ. If a hexagon $H \in \mathcal{H}_\delta$ intersects A, we color it white, and if it meets the boundary but not A, we color it black. For each hexagon in the interior, color it black or white with probability $\frac{1}{2}$ independently. The percolation boundary is defined to be the interface γ between the black and white regions connecting a to b.

It has been shown (among other things) that, in a certain precise sense that we will not describe, the $\delta \to 0$ limit of the law of γ exists and is conformally invariant. By conformally invariant we mean that if $G : D \to D'$ is a conformal (bijective and holomorphic in both directions) map sending a, b to a', b', then the laws are related by $\mu_{Dab} = \mu_{D'a'b'} \circ G$, where μ_{Dab} is the limit law in domain D from a to b. This is not too surprising, since this is closely related to the following process.

- Start a random walk on $\mathcal{H}_\delta$ at some vertex in D.
- At each step, with probability $\frac{1}{2}$, turn either right or left and move forward in that direction. Moving backwards is not allowed.
- Stop when the walk hits the boundary of D.

This has Brownian motion as its scaling limit, which is itself conformally invariant. We can now us make a simplifying assumption. By the Riemann mapping theorem and Carathéodory's theorem,

Theorem 1.1. *(Riemann mapping theorem) If $D, D' \subsetneq \mathbb{C}$ are simply connected domains, then there exists a conformal map $G : D \to D'$. Moreover, if $x \in D$, $y \in D'$, and $\theta \in [0, 2\pi)$ and we assert that $G(x) = y$ and $\arg G'(x) = \theta$, then the map is unique.*

Theorem 1.2. *(Carathéodory's theorem) If $D, D' \subsetneq \mathbb{C}$ are simply connected domains bounded by a Jordan curve, then any conformal map between the extends to a homeomorphism $\bar{D} \to \bar{D}'$.*

Remark 1.3. *In general, it is not true that conformal maps extend to homeomorphisms of the topological boundaries, even on the Riemann sphere $\bar{\mathbb{C}}$. However, this result does hold if we replace the topological boundary with the Martin boundary (see [BN16]), and percolation can also be defined in this setting.*

such a map exists and we may restrict our attention to critical percolation from 0 to ∞ in the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. Assume that A is to the right of the origin. Consider the following property:

Proposition 1.4. *(Markov-type property for percolation) Conditioned on the event that the first k steps trace out a discrete curve β, the law of $\gamma \backslash \beta$ is that of a percolation interface from the terminal point of β to ∞ with boundary $\partial D \cup \hat{\beta}$, where $\hat{\beta}$ is an appropriate small neighborhood of β.*

This follows from the fact that there are black hexagons to the left and white hexagons to the right of β, and from independence of the colors of hexagons in the interior. We had to consider a small neighborhood of β only because we are defining percolation on a domain with a Jordan curve as boundary, which again could be avoided by carefully restating the model. This proposition leads to the idea that the limiting curve has a domain Markov property.

Definition 1.5. *(Informal domain Markov property) Conditioned on $\gamma[0,t]$, the law of $\gamma[t,\infty)$ is that of $\mu_{H_t \gamma_t \infty}$, where H_t is the unbounded component of $\mathbb{H} \backslash \gamma[0,t]$.*

This, along with conformal invariance, are the motivating properties for SLE. An important idea in the analysis of laws which may satisfy these properties is to consider a family of conformal maps $g_t : H_t \to \mathbb{H}$, so that the domain Markov property becomes a statement about SLE on $\mathbb{H}$, without reference to the potentially complicated H_t. It may seem natural to normalize g_t so that it sends γ_t to 0, and then it would say that $g_t(\gamma[t,\infty))$ is independent of $\gamma[0,t]$ and has the same law as γ. While this is true, it will be easier to normalize in a more intrinsic way which does not depend on the terminal point of $\gamma[0,t]$. This is what we will describe in what follows. From this point, we adopt a more rigorous style.

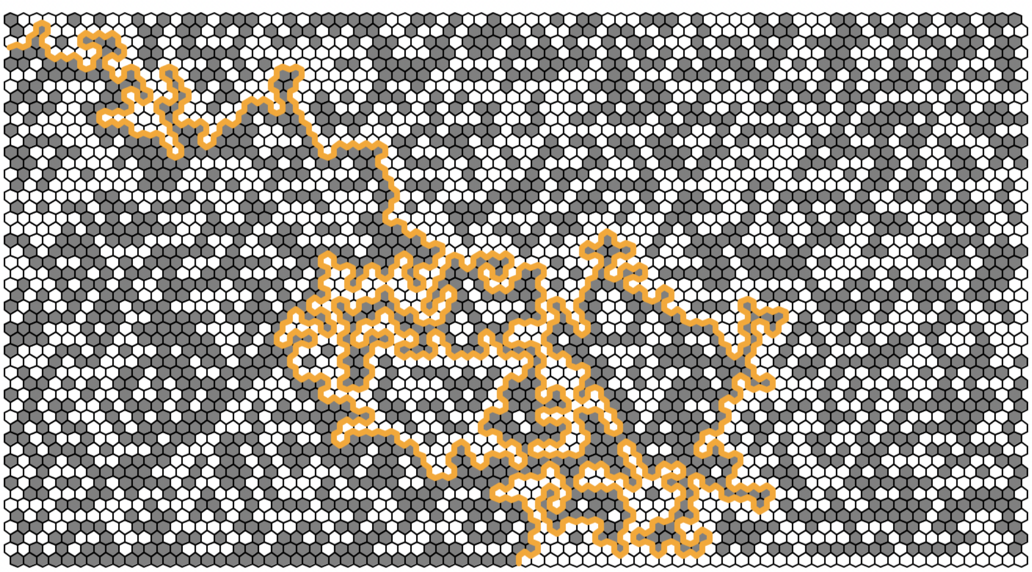

Figure 1. Realization of critical percolation on $\mathbb{H}$

1.2. **Outline.** In section 2, we introduce the objects of Loewner's theory. In section 3, we construct the Schramm Loewner evolution and discuss some properties. In section 3.1, we come back to the critical percolation problem and give a heuristic argument for why the limit must be SLE(6).

2. Loewner Theory

Definition 2.1. *($\mathbb{H}$-hull) A subset $K \subset \mathbb{H}$ is said to be an $\mathbb{H}$-hull if it is bounded and $\mathbb{H}\backslash K$ is a simply connected domain.*

Definition 2.2. *($\mathbb{H}$-hulls of a curve) The family of $\mathbb{H}$-hulls $(K_t)_{t\geqslant 0}$ generated by a continuous curve $(\gamma_t)_{t\geqslant 0}$ in $\bar{\mathbb{H}}$ is defined to be such that K_t is the complement of the unbounded component of $\mathbb{H}\backslash\gamma[0,t]$.*

Proposition 2.3. *(Mapping out function) Let K be an $\mathbb{H}$-hull. There exists a unique conformal map $g_K : \mathbb{H}\backslash K \to \mathbb{H}$ such that*

$$g_K(z) - z \to 0 \quad \textit{as} \quad t \to \infty.$$

We call this the mapping out function of K.

Proof. By the Riemann mapping theorem, there exists some conformal map $g : \mathbb{H}\backslash K \to \mathbb{H}$. Suppose K is contained in $B_r(0)$ and denote $A = \{|z| > r\}$. Since the points on $(B \cap \mathbb{R}) \cup \infty$ are all simple boundary points on $\bar{\mathbb{C}}$ (see [Con12]), g extends continuously to a homeomorphism $g : (A \cap \mathbb{R}) \cup \infty \to \mathbb{R} \cup \infty$. By a rotation, we can force $g(\infty) = \infty$ and $g(x) \in \mathbb{R}$ whenever $x \in A \cap \mathbb{R}$. By the Schwarz reflection principle, it follows that g extends to a conformal transformation of A. Since it is injective, there is a simple pole at infinity and we have the following expansion:

$$g(z) = a_{-1}z + a_0 + a_1 z^{-1} + O(|z|^{-2}).$$

Since g maps $A \cap \mathbb{R}$ to $\mathbb{R}$, all coefficients are real. Since g maps $\mathbb{H}$ to $\mathbb{H}$, the first coefficient is positive. The only conformal automorphisms of $\mathbb{H}$ which send infinity to infinity are of the form $z \mapsto az + b$ for $a > 0$ and $b \in \mathbb{R}$, so we construct our function uniquely as $g_K = \frac{1}{a_{-1}}g - \frac{a_0}{a_{-1}}$. □

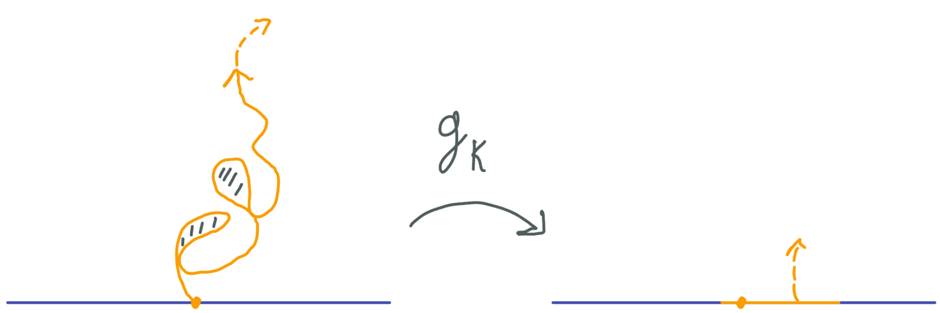

Figure 2. Mapping out function corresponding to the hull of a non-self-crossing curve.

The above will be the correct normalization for the conformal maps we are looking for. The idea will be to start with these maps and get the SLE curve out of them. Next we will discuss the correct parametrization of these maps by capacity.

Definition 2.4. *(Half plane capacity) The half plane capacity of an $\mathbb{H}$-hull K, denoted hcap(K), is said to be the quantity that satisfies*

$$g_K(z) = z + \frac{hcap(K)}{z} + O(|z|^{-2})$$

as $z \to \infty$, where g_K is the same mapping out function discussed above.

Definition 2.5. *(Parametrization by capacity) We say that a family of hulls $(K_t)_{t\geqslant 0}$ is parametrized by capacity if hcap$(K_t) = 2t$. We say a curve is parametrized by capacity if its hulls are parametrized by capacity.*

Proposition 2.6. *(Scaling property of hcap) For any $\mathbb{H}$-hull K and $\lambda > 0$, we have the scaling property hcap$(\lambda K) = \lambda^2$hcap(K).*

Proof. Note that $\lambda \circ g_K \circ \lambda^{-1}$ has the following expansion at infinity:

$$\lambda(\lambda^{-1}z) + \frac{\lambda \text{hcap}(K)}{\lambda^{-1}z} + \ldots = z + \frac{\lambda^2 \text{hcap}(K)}{z} + \ldots.$$

Therefore, it is the mapping out function of λK and the result follows. □

The reason for choosing this parametrization is the following theorem. We remark that similar results hold if we choose any λt instead of $2t$; this is merely convention.

Theorem 2.7. *(Loewner's equation) Take $t \mapsto \xi_t$ a continuous real valued funciton. For each $z \in \mathbb{H}$, let $g_t(z)$ denote the solution to the initial value problem*

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \xi_t}, \quad g_0(z) = z.$$

Let τ_z be the supremum over $t \geqslant 0$ such that the solution is defined up to time t with $g_t(z) \in \mathbb{H}$. Let $K_t = \{z \in \mathbb{H} : \tau_z \leqslant t\}$. Then g_t is the unique conformal map from $\mathbb{H}\backslash K_t$ to $\mathbb{H}$ such that $g_t(z) - z \to 0$ as $z \to \infty$. Moreover, hcap$(K_t) = 2t$.

A version of the converse of this theorem is also true: if we have a non-self-crossing curve γ_t, we may reparametrize it by capacity, and then the corresponding mapping out functions satisfy the same differential equation. See [Law08] for more details on both directions. Our task is therefore to pick the correct (random) driving function ξ_t.

Definition 2.8. *In this setup, we say that*

(1) *$(g_t)_{t\geqslant 0}$ is the chordal Loewner evolution (or flow or chain) with driving function $(\xi_t)_{t\geqslant 0}$*
(2) *$(\xi_t)_{t\geqslant 0}$ is the Loewner transform of $(K_t)_{t\geqslant 0}$*
(3) *K_t is the hull of the Loewner evolution at time t*
(4) *$(g_t^{-1}(\xi_t))_{t\geqslant 0}$ is the path of the Loewner evolution, if it is continuous and generates $(K_t)_{t\geqslant 0}$.*

The caveat in the last point is important. Not every real valued continuous driving function yields a continuous curve. There are several criteria for the existence of such a curve which come from Loewner's theory. For example, if the driving function has sufficiently small Hölder-$\frac{1}{2}$ norm, then the hulls are generated by a simple curve. However, this will not be enough for our purposes.

3. Schramm Loewner Evolution

As we have seen, it is more natural to deal with hulls and conformal maps instead of curves. Therefore, we will restate the domain Markov property and a consequence of conformal invariance, scaling invariance, in terms of these objects. To do this, we must first define a measure space of "continuously growing" families of $\mathbb{H}$-hulls. This will let us talk about random variables in this space.

Write $\mathcal{K}$ for the set of $\mathbb{H}$-hulls, and fix a metric d of uniform convergence on $C(\mathbb{H}, \mathbb{H})$. The space $\mathcal{K}$ admits the metric

$$d_{\mathcal{K}}(K_1, K_2) = d(g_{K_1}^{-1}, g_{K_2}^{-1}),$$

called the Carathéodory metric. The families we are interested in are hulls that can arise from Loewner's equation, which are all elements of some set $\mathcal{L} \subset C([0, \infty), \mathcal{K})$ parametrized by capacity. For an alternative characterization, see [BN16]. We make this into a measure space by taking the σ-algebra to be the Borel sets of the topology of uniform convergence on compact time intervals.

Definition 3.1. *(Scale Invariance) Define $K_t^\lambda = \lambda K_{\lambda^{-2}t}$. We say that a random variable $(K_t)_{t\geqslant 0} \in \mathcal{L}$ is scale invariant if it has the same law as $(K_t^\lambda)_{t\geqslant 0}$ (which is also an element of $\mathcal{L}$ by proposition 2.6) for any $\lambda > 0$.*

Definition 3.2. *(Domain Markov Property) Define $K_t^{(s)} = g_{K_s}(K_{s+t} \backslash K_s) - \xi_s$. We say that a random variable $(K_t)_{t\geqslant 0} \in \mathcal{L}$ has the domain Markov property if $(K_t^{(s)})_{t\geqslant 0}$ has the same law and is independent of $\sigma(\xi_r : r \leqslant s)$ for any $s \geqslant 0$.*

These two properties are enough to uniquely determine the (random) driving function we are after. Scale invariance translates to "ξ_t is invariant under brownian scaling." The domain Markov property translates to "ξ_t has stationary independent increments." Together with continuity, these imply that ξ_t is a Brownian motion with diffusivity $\kappa \geqslant 0$.

Definition 3.3. *(Schramm Loewner Evolution) The chordal Schramm-Loewner evolution from 0 to infinity in $\mathbb{H}$ with parameter $\kappa \geqslant 0$ (or just chordal SLE(κ)) is the chordal Loewner evolution driven by $\xi_t = \sqrt{\kappa} B_t$, where B_t is a standard Brownian motion.*

See [Law08] for the radial version of the Schramm Loewner evolution. As we discussed in the previous section, it does not readily follow that Schramm Loewner evolutions are generated by curves. The following is therefore a nontrivial result. See [Law08] for a proof of the $\kappa \neq 8$ case.

Theorem 3.4. *(Rhode-Schramm) The map g_t^{-1} extends continuously to ξ_t for all $t \geqslant 0$, almost surely. Moreover, $t \mapsto \gamma_t = g_t^{-1}(\xi_t)$ is continuous and generates $(K_t)_{t\geqslant 0}$, almost surely. We call γ_t the SLE curve.*

Interestingly, SLEs exhibit a wide range of different behaviors for different values of κ. The following is the canonical example of this.

Theorem 3.5. *(Phases of SLE) Let $(\gamma_t)_{t\geqslant 0}$ denote the curve of an SLE(κ), and let $(K_t)_{t\geqslant 0}$ be its hulls.*

(1) (Simple phase) If $\kappa \in [0, 4]$ then γ_t is almost surely a simple curve.
(2) (Swallowing phase) If $\kappa \in (4, 8)$ then $\bigcup_{t\geqslant 0} K_t = \mathbb{H}$ almost surely.
(3) (Space filling phase) If $\kappa \in [8, \infty)$ then $\gamma[0, \infty) = \bar{\mathbb{H}}$ almost surely.

For a proof of the first two, we refer the reader to [BN16]. The basic idea is to relate SLE(κ) to the more familiar Bessel process with parameter $\frac{2}{\kappa}$.

3.1. **Locality of SLE(6).** By theorem 3.5, we can already guess that the limit of the critical percolation interface should be an SLE with parameter $\kappa \in (4, 8)$. However, we can say more. Intuitively, this interface should not know anything about the domain it is in, other than the fact that it should turn to infinity whenever it hits the boundary. This can be stated rigourously in the discrete case, and we leave it as an exercise. In the continuous setting, it would look something like the next theorem. This is an important property, which also what lets us prove things about Brownian motion as in [MP01].

Theorem 3.6. *(Locality of SLE(6)) Let $N, \tilde{N} \subset \mathbb{H}$ be simply connected domains and let $I, \tilde{I} \subset \mathbb{R}$ be open intervals containing 0 such that $N, \tilde{N}$, respectively, are neighborhoods in $\mathbb{H}$. Let $\phi : N \cup I \to \tilde{N} \cup \tilde{I}$ be a homeomorphism such that $\phi(0) = 0$ which restricts to a conformal map $N \to \tilde{N}$. Let $(\gamma_t)_{t \geqslant 0}$ be an SLE(6) curve and consider the stopping times*

$$T = \inf\{t \geqslant 0 : \gamma_t \notin N \cup I\}, \quad \tilde{T} = \inf\{t \geqslant 0 : \gamma_t \notin \tilde{N} \cup \tilde{I}\}.$$

Then $(\phi(\gamma_t))_{t<T}$ in its reparametrization by capacity and $(\gamma_t)_{t<\tilde{T}}$ are equidistributed.

For the sake of exposition, we will abstract away most of the complex analysis. Firstly, if I and $\tilde{I}$ are proper subsets of $\mathbb{R}$, then such a ϕ must exist, and it is in fact unique. This can be shown by a reflection argument. Secondly, we construct maps with several nice properties.

Lemma 3.7. *Let $(K_t) \in \mathcal{L}$, and write ξ_t for its Loewner transform. Take ϕ be as in theorem 3.6, and let $\tilde{K}_t = \phi(K_t)$ with Loewner transform $\tilde{\xi}_t$. Then the maps*

$$\phi_t = g_{\tilde{K}_t} \circ \phi \circ g_{K_t}^{-1} : \mathbb{H} \to \mathbb{H}$$

have the following properties:

$$\tilde{\xi}_t = \phi_t(\xi_t), \quad hcap(\tilde{K}_t) = 2\int_0^t \phi_s'(\xi_s)^2 ds,$$

$$(t, x) \mapsto \phi_t(x) \in C^{1,2} \text{ on an open set containing } (t, \xi_t) \text{ for } t < T,$$

$$\dot{\phi}(t) = -3\phi''(t) \quad \forall\, t < T.$$

This lemma is proven in [BN16]. We can now prove the main theorem of this section.

Proof. (Locality of SLE(6)) By the previous lemma and the generalized Itô formula as in [Law08],

$$\begin{aligned} d\tilde{\xi}_t^{\,T} &= \dot{\phi}_t(\xi_t^T)dt + \phi_t'(\xi_t^T)d\xi_t^T + \frac{1}{2}\phi_t''(\xi_t^T)d\langle\xi^T\rangle_t \\ &= (-3\phi_t''(\xi_t^T) + \frac{\kappa}{2}\phi_t''(\xi_t^T))dt + \phi_t'(\xi_t^T)d\xi_t^T. \end{aligned}$$

In our case $\kappa = 6$, so the finite variation term vanishes and we are left with a local martingale with quadratic variation equal to

$$\langle\tilde{\xi}^T\rangle_t = \int_0^t \phi_t'(\xi_t^T)^2 d(6t) = 3\text{hcap}(\tilde{K}_t) \quad \forall\, t < T.$$

Let $\eta_s = \tilde{\xi}_{\tau_s}$ be its parametrization by capacity, and let S be the time-changed stopping time. By the optional stopping theorem, η_s is still a continuous local martingale with respect to its filtration. It has quadratic variation equal to

$$\langle\eta^S\rangle_s = \langle\tilde{\xi}^S\rangle_{\tau_s} = 3 \cdot 2s = 6s \quad \forall\, s < S.$$

By Lévy's characterization, it follows that $(\eta_s)_{s<S}$ extends to a Brownian motion $(\eta_s)_{s\geqslant 0}$ of diffusivity 6. Write $\tilde{\gamma}$ for the SLE(6) curve driven by $(\eta_s)_{s\geqslant 0}$, so $\phi(\gamma_{\tau_s}) = \tilde{\gamma}_s$ whenever $s < S$ and $S = \inf\{s \geqslant 0 : \tilde{\gamma}_s \notin \tilde{N} \cup \tilde{I}\}$. Thus $(\phi(\gamma_{\tau_s}))_{s<S}$ and $(\gamma_s)_{s<\tilde{T}}$ are equidistributed, as desired. □

This suggests Smirnov's result; the limit of the critical percolation interface is an SLE(6) curve. The vanishing of the finite variation term which occurs in the above proof only occurs for this value of κ.

4. Conclusions

In this article, we motivated the chordal Schramm Loewner evolution as the scaling limit of the critical percolation interface. This gave three key properties: conformal invariance, the domain Markov property, and locality. We found that there is a unique family of curves satisfying the first two properties and called them curves of the chordal Schramm Loewner evolution. If we restrict to the upper half plane, uniqueness is true if we assert that the curve is scale invariant, instead of conformally invariant. To do this, we developed Loewner's theory and considered families of hulls and of conformal maps, and SLE was the curve resulting from the Loewner differential equation with Brownian motion as its driving function. After we added the locality property back in, we saw that the only such curve is SLE(6), which suggests that this should be the scaling limit of our motivating problem.

We mentioned that SLE comes up as a scaling limit of several statistical physics model. Here is a list of a few of them, with their corresponding parameter values. The ones marked with (*) are conjectured scaling limits. In most of them, the results are conjectured to generalize further. The layout is left as a puzzle.

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- -
- $\kappa = 2$: Loop-erased random walk
- $\kappa = \frac{8}{3}$: Self-avoiding walk (*)
- $\kappa = 3$: Ising model
- $\kappa = 4$: Harmonic explorer

- $\kappa = 16$: Schnyder woods (*)
- $\kappa = 12$: Bipolar orientations (*)
- $\kappa = 8$: Uniform spanning tree
- $\kappa = 6$: Critical percolation
- $\kappa = \frac{16}{3}$: FK Ising model
- $\kappa = 4$: Level lines of Gaussian free field

(Credits to Professor Scott Sheffield; this list resulted from my understanding of his lecture).

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